Appendix LL

Algebraic Solution of the Oscillator

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

Sidney Coleman

We now return to the harmonic oscillator using operator methods to obtain the wave functions and the energy. In Chapter 12, the harmonic oscillator is used as a model for the quantum mechanic treatment of the electromagnetic field. An expression for the energy of the harmonic oscillator is obtained by adding the kinetic energy $p^2/2m$ to the general expression for the potential energy of the oscillator provided by Eq. (I.4) to obtain

$$E = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2.$$

Following the procedure described in Chapter 3, the energy operator is obtained by replacing the momentum p and the position x in this last equation by the operators, $\hat{p}$ and $\hat{x}$, respectively. We obtain

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2. \tag{LL.1}$$

We recall that the momentum operator is given by Eq. (DD.1) and the position operator $\hat{x}$ is equal to the position coordinate x.

To see how we might find the wave functions for the harmonic oscillator by algebraic methods, we first note that if the momentum operator $\hat{p}$ was an ordinary number, we could factor the Hamiltonian operator for the oscillator given by eq. (LL.1) as follows:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{\mathrm{i}\hat{p}}{m\omega}\right)\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{\mathrm{i}\hat{p}}{m\omega}\right). \tag{LL.2}$$

This factorization may readily be confirmed using the identity

$$(a - \mathrm{i}b)(a + \mathrm{i}b) = a^2 + b^2.$$

Such a factorization of the Hamiltonian cannot be carried out because the momentum and the position are represented in quantum mechanics by operators, and the order of two operators cannot generally be interchanged as can the products of numbers.

The effect of changing the order of the momentum and position operators can be determined by using the explicit form of the momentum operator given by Eq. (DD.1). The product of the momentum and position operators is

$$\hat{p}\hat{x} = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}x.$$

We can study the properties of this operator product by allowing it to act on an arbitrary wave function $\psi(x)$ giving

$$-i\hbar\frac{d}{dx}x\psi = -i\hbar\frac{d}{dx}(x\psi)\ .$$

The derivative of $(x\psi)$ on the right can now be evaluated using the ordinary product rule to obtain

$$-i\hbar\frac{d}{dx}(x\psi) = -i\hbar\psi - i\hbar x\frac{d\psi}{dx}.$$

Bringing the last term on the right-hand side of the above equation over to the left-hand side and multiplying the whole equation through with -1, we obtain

$$\left(-i\hbar x\frac{d}{dx} + i\hbar\frac{d}{dx}x\right)\psi = i\hbar\psi.$$

Since this last equation is true for any arbitrary function ψ, we can write it as an operator identity

$$-i\hbar x\frac{d}{dx} + i\hbar\frac{d}{dx}x = i\hbar$$

or

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{LL.3}$$

The expression on the left-hand side of this last equation involving the position and momentum operators, $\hat{x}$ and $\hat{p}$, may be written more simply using the idea of a commutator. The *commutator* of two operators, A and B, is defined by the following equation:

$$[A, B] = AB - BA. \tag{LL.4}$$

Using this notation, Eq. (LL.3) becomes

$$[\hat{x}, \hat{p}] = i\hbar. \tag{LL.5}$$

Even though the Hamiltonian operator (LL.1) cannot be factored as simply as indicated in Eq. (LL.2), we would like to evaluate the product of the "factors" occurring in this equation. For this purpose, we define the two operators,

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \tag{LL.6}$$

and

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right). \tag{LL.7}$$

We consider now the operator product

$$\hbar\omega a^\dagger a, \tag{LL.8}$$

which is equal to the expression appearing after the second equality in Eq. (LL.2).

Using the commutation relation (LL.5), the operator product (LL.8) can be written as

$$\begin{aligned}\hbar\omega a^\dagger a &= \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}[\hat{x}, \hat{p}]\right)\\ &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{1}{2}\hbar\omega.\end{aligned} \tag{LL.9}$$

The first two terms appearing after the second equality can be identified as the oscillator Hamiltonian $\hat{H}$ given by Eq. (LL.1). Bringing the last term on the right-hand side of Eq. (LL.9) over to the left-hand side and then interchanging the two sides of the equation, we obtain

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{LL.10}$$

This equation may be regarded as the corrected version of the naive factorization (LL.2).

We note that the order of the operators $a^\dagger$ and a in Eq. (LL.9) is important. The same line of argument with the operators, $a^\dagger$ and a, interchanged leads to the following equation:

$$\hbar\omega\, aa^\dagger = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}\hbar\omega. \tag{LL.11}$$

We can obtain a commutation relation for the operators a and $a^\dagger$ by subtracting Eq. (LL.9) from Eq. (LL.11) giving

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1. \tag{LL.12}$$

The commutation relation between a and $a^\dagger$ may be used to derive a relation between successive eigenfunctions of the oscillator Hamiltonian. Suppose that the wave function ψ is an eigenfunction of the Hamiltonian (LL.10) corresponding to the eigenvalue E. Then ψ satisfies the following equation:

$$\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\psi = E\psi. \tag{LL.13}$$

Multiplying this equation from the left with $a^\dagger$ gives

$$\hbar\omega\left(a^\dagger a^\dagger a + \frac{1}{2}a^\dagger\right)\psi = Ea^\dagger\psi.$$

The first term on the left-hand side of this last equation may be rewritten by using the commutation relation (LL.12) to replace $a^\dagger a$ with $aa^\dagger - 1$ giving

$$\hbar\omega\left(a^\dagger a - 1 + \frac{1}{2}\right)a^\dagger\psi = Ea^\dagger\psi.$$

Finally, we take the second term on the left-hand side over to the right-hand side to obtain

$$\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a^\dagger\psi = (E + \hbar\omega)a^\dagger\psi.$$

Hence, the wave function $a^\dagger\psi$ is an eigenfunction of the oscillator Hamiltonian corresponding to the eigenvalue $E + \hbar\omega$. We shall thus refer to $a^\dagger$ as a *raising operator* or a *step-up operator*. It transforms an eigenfunction of $\hat{H}$ into an eigenfunction corresponding to the next higher eigenvalue. In the same way, the operator a may be shown to be a *lowering operator* or *step-down operator* which transforms an eigenfunction of $\hat{H}$ into an eigenfunction corresponding to the next lower eigenvalue. The effect of the raising and lowering operators on the states of the Harmonic oscillator is illustrated in Fig. LL.1.

By repeatedly operating on an eigenfunction of the harmonic oscillator with the lowering operator, one can produce eigenfunctions corresponding to lower and lower eigenvalues. Allowed to continue indefinitely, this process would eventually lead to energy eigenvalues less than zero. However, one may show that the harmonic oscillator, which has a potential energy that is positive everywhere, cannot have a bound state with a negative eigenvalue.

The lowering process must end in some way. This can only happen by the product of the lowering operator a and the wave function producing a zero function. Denoting the lowest bound state by ψ_0, we must have

$$a\psi_0 = 0. \tag{LL.14}$$

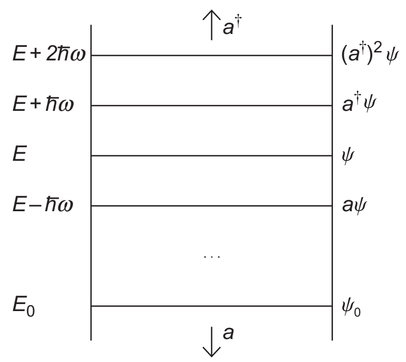

FIGURE LL.1 The effect of the raising and lowering operators on the states of the harmonic operator.

Since every term in an eigenvalue equation depends upon the eigenfunction, an eigenvalue equation will always be satisfied by a function which is equal to zero everywhere.

The energy of the lowest state can be determined by substituting the function ψ_0 into Eq. (LL.13) to obtain

$$\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\psi_0 = E_0\psi_0.$$

Since $a\psi_0 = 0$, the first term on the left-hand side of this equation must be equal to zero. We must have

$$E_0 = \frac{1}{2}\,\hbar\omega.$$

A state of the oscillator with $\hbar\omega$ more energy can be obtained by operating with the operator $a^\dagger$ upon the lowest state, and this process can be continued producing states with greater and greater energy. The energy levels of the oscillator are thus given by the formula

$$E = \hbar\omega(n + 1/2).$$

Equation (LL.14) can be solved for the wave function corresponding to the lowest eigenvalue. More energetic states can then be obtained by operating upon ψ_0 with the step-up operator $a^\dagger$. Substituting Eq. (LL.7) into Eq. (LL.14), we obtain

$$\left(\hat{x} + \frac{\mathrm{i}\hat{p}}{m\omega}\right)\psi_0 = 0.$$

This equation may be written as a differential equation by using the explicit expression for the momentum operator given by Eq. (DD.1) to get

$$\frac{\hbar}{m\omega}\frac{\mathrm{d}\psi_0}{\mathrm{d}x} + x\psi_0 = 0$$

or

$$\frac{\mathrm{d}\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}x\,\mathrm{d}x.$$

Integrating this last equation, we obtain

$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + \ln A_0,$$

where we have denoted the arbitrary integration constant as $\ln A_0$. Taking the term $\ln A_0$ over to the left-hand side of the equation and using the properties of the natural logarithm, we obtain

$$\ln\left(\frac{\psi_0}{A_0}\right) = -\frac{m\omega}{2\hbar}x^2.$$

The lowest eigenfunction ψ_0 can then be obtained by taking the anti-logarithm of both sides of the equation and rearranging terms to get

$$\psi_0 = A_0\mathrm{e}^{(-m\omega/2\hbar)x^2}. \tag{LL.15}$$

All of the results obtained in the section have been obtained before by solving the Schrödinger equation for the oscillator. The energy and the wave function of the oscillator are given by Eqs. (2.44) and (2.45). We leave as an exercise to use the methods developed in this section to obtain the wave functions of the first two excited states of the oscillator. The algebraic solution of this problem provides its own insights into the oscillator. Since the harmonic oscillator can only vibrate with a single frequency, all of its states can be regarded as excitations of a single state. The first excited state of the oscillator corresponds to a state for which the oscillator has absorbed a single photon with energy $\hbar\omega$, and higher excited states correspond to states for which the oscillator has absorbed additional photons. This way of thinking about the oscillator will be used in Chapter 12 as a model for the quantization of the electromagnetic field.